MATH 54 – HINTS TO HOMEWORK 10

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Here are a couple of hints to Homework 10! Enjoy :)

SECTION 6.1: INNER PRODUCTS, LENGTHS, AND ORTHOGONALITY

6.1.9. Calculate $\frac{\mathbf{u}}{\|\mathbf{u}\|}$

6.1.13. Calculate $\|\mathbf{x}-\mathbf{y}\|$

6.1.15. Check if $\mathbf{a}\cdot\mathbf{b}=0$ or not

6.1.19.

(a) **T**
(b) **T**
(c) **T** (either look on page 319, or do it directly: you're given that $\|\mathbf{u}-\mathbf{v}\|=\|\mathbf{u}+\mathbf{v}\|$, Now square this to get $(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})=(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$, expand this out and cancel out terms until you eventually get $\mathbf{u}\cdot\mathbf{v}=0$)
(d) **F** (those two concepts are unrelated! For example, consider $A=\begin{bmatrix}1&0&0\\1&0&0\\0&0&0\end{bmatrix}$, then $Col(A)=Span\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$, while $Nul(A)=Span\left\{\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$. Then, for example $\begin{bmatrix}1\\1\\0\end{bmatrix}$ is not orthogonal to $\begin{bmatrix}0\\1\\0\end{bmatrix}$. However, what is true is that $Nul(A)$ is perpendicular to $Row(A)$
(e) **T** (Remember $W^{\perp}$ is the set of vectors orthogonal to W. Now if $\mathbf{x}$ is orthogonal to each $\mathbf{v_j}$, then $\mathbf{x}$ is orthogonal to W because the $\mathbf{v_j}$ span W)

6.1.20.

(a) **T** ($\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$)
(b) **F** (Take $\mathbf{v}=\begin{bmatrix}1\\0\end{bmatrix}$ and $c=-1$)
(c) **T** (by definition of $W^{\perp}$)
(d) **T** (expend the right-hand-side out and cancel some terms)
(e) **T** (See theorem 3 on page 321. Don't worry too much about that for the exam)

6.1.22. $\mathbf{u}\cdot\mathbf{u}=0$ only if $\mathbf{u}=\mathbf{0}$

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6.1.24. Use the fact that $\|\mathbf{u}+\mathbf{v}\| = (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v})$ and expand this out! Similar for the other term! Also, use the fact that $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ and similar for the other term!

6.1.27. Calculate $\mathbf{y} \cdot (\mathbf{u}+\mathbf{v})$

SECTION 6.2: ORTHOGONAL SETS

Remember: A set $\mathcal{B}$ is orthogonal if for every pair of distinct vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v} = 0$. It is orthonormal if it is orthogonal and every vector has length 1. An orthogonal set can be made orthonormal by dividing every vector by its length.

6.2.1, 6.2.3. Show that $\mathbf{u} \cdot \mathbf{v} = 0$ (or not), $\mathbf{u} \cdot \mathbf{w} = 0$ (or not) and $\mathbf{u} \cdot \mathbf{w} = 0$ (or not)

6.2.7. Show $\mathbf{u_1} \cdot \mathbf{u_2} = 0$, Then use the fact that if $\mathbf{x} = a\mathbf{u_1} + b\mathbf{u_2}$, then $a = \frac{\mathbf{x}\cdot\mathbf{u_1}}{\mathbf{u_1}\cdot\mathbf{u_1}}$, and $b = \frac{\mathbf{x}\cdot\mathbf{u_2}}{\mathbf{u_2}\cdot\mathbf{u_2}}$

6.2.9. Similar to 6.2.7

6.2.11, 6.2.15. The formula for orthogonal projection of $\mathbf{y}$ on the line spanned by $\mathbf{u}$ is:

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

The distance between $\mathbf{u}$ and L is then $\|\mathbf{y} - \hat{\mathbf{y}}\|$

6.2.17, 6.2.19. Similar to 6.2.1, 6.2.3. In addition, you have to verify that each vector has length 1. If it doesn't, calculate $\frac{\mathbf{u}}{\|\mathbf{u}\|}$

6.2.23.

(a) **T** (Consider $\left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix} \right\}$. It is linearly independent, but not orthogonal)
(b) **T**(just use dot products / hugging)
(c) **F**
(d) T
(e) **F** (it's $\|\mathbf{y} - \hat{\mathbf{y}}\|$)

6.2.24.

(a) **T**(it could contain the **0** vector and still be orthogonal, but if you ignore this, it is F)
(b) **F** (the length of each vector has to be $= 1$)
(c) **T**($\|A\mathbf{x}\| = \|\mathbf{x}\|$)
(d) **T**(This is because $\left(\frac{\mathbf{y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{\mathbf{y}\cdot c\mathbf{v}}{c\mathbf{v}\cdot c\mathbf{v}}\right)c\mathbf{v}$)
(e) **T**(it has determinant ± 1)

6.2.29. First of all, UV is invertible because $det(UV) = det(U)det(V) = (\pm1)(\pm1) = \pm1 \neq 0$, also $(UV)^{-1} = V^{-1}U^{-1} = V^TU^T = (UV)^T$

SECTION 6.3: ORTHOGONAL PROJECTION

Here are all the basic facts that you'll need:

(1) If $W = Span\{\mathbf{u_1}, \mathbf{u_2} \cdots \mathbf{u_k}\}$, then the orthogonal projection of $\mathbf{y}$ onto W is:

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\right)\mathbf{u_1} + \left(\frac{\mathbf{y} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}}\right)\mathbf{u_2} + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{u_k}}{\mathbf{u_k} \cdot \mathbf{u_k}}\right)\mathbf{u_k}$$

(2) Then $\hat{\mathbf{y}}$ is in W, $y - \hat{\mathbf{y}}$ is in $W^{\perp}$ (that is, orthogonal to W).

(3) $\mathbf{y} = (\hat{\mathbf{y}}) + (y - \hat{\mathbf{y}})$, which decomposes $\mathbf{y}$ as a sum of two vectors, one in W and the other one orthogonal to W.

(4) $\hat{\mathbf{y}}$ is the closest point to $\mathbf{y}$ in W.

(5) $\|y - \hat{\mathbf{y}}\|$ is the smallest distance between $\mathbf{y}$ and W.

6.3.1. $\mathbf{x} = \hat{\mathbf{x}} + (\mathbf{x} - \hat{\mathbf{x}})$, where $\hat{\mathbf{x}}$ is the projection on $Span\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$. Use the formula above!

6.3.3, 6.3.5, 6.3.7, 6.3.18(b). $\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}}\right)\mathbf{u_1} + \left(\frac{\mathbf{y} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}}\right)\mathbf{u_2}$. For 6.3.7, they mean $\mathbf{y} = \hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})$, and for 6.3.11, they mean $\hat{\mathbf{y}}$

6.3.21.

(a) **T**

(b) **T**

(c) **F** (see the sentence after the end of the proof on page 336)

(d) **T**

(e) **T**

6.3.22.

(a) **T**(if $\mathbf{v}$ is in $W^{\perp}$, then $\mathbf{v} \cdot \mathbf{w} = 0$ for every $\mathbf{w}$ in W. But if $\mathbf{v}$ is also in W, then we can let $\mathbf{w} = \mathbf{v}$, so $\mathbf{v} \cdot \mathbf{v} = 0$, but then $\mathbf{v} = \mathbf{0}$)

(b) **T**(It's the sum of the orthogonal projections on $Span\{\mathbf{v_1}\}$, $Span\{\mathbf{v_2}\}$, etc., see page 339)

(c) **T**(by uniqueness of such a decomposition)

(d) **T**

(e) **F** (It's $U^T U\mathbf{x} = \mathbf{x}$)

SECTION 6.4: THE GRAM-SCHMIDT PROCESS

Gram-Schmidt process: Let's say you want to find an orthogonal basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ from $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$

First let $\mathbf{v_1} = \mathbf{u_1}$ and cross out $\mathbf{u_1}$ from your list!
Then calculate $\hat{\mathbf{u_2}} = \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1}$.

Then let $\mathbf{v_2} = \mathbf{u_2} - \hat{\mathbf{u_2}}$, and cross out $\mathbf{u_2}$ from your list!

If you're given only 2 vectors, you're done, otherwise calculate:
$\hat{\mathbf{u_3}} = \left(\frac{\mathbf{u_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} + \left(\frac{\mathbf{u_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right)\mathbf{v_2}$

Then let $\mathbf{v_3} = \mathbf{u_3} - \hat{\mathbf{u_3}}$, and cross out $\mathbf{u_3}$ from your list!

If you're given only 3 vecotrs, you're done, otherwise repeat this process until you run out of vectors in your list!

To get an *orthonormal* basis, just divide every vector at the end by its length. At every step, it's helpful to multiply your vector by a scalar to avoid fractions. This is ok, because you'll normalize them at the end anyway!

6.4.9, 6.4.11. What they mean is apply Gram-Schmidt to $\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}$, similar with $6.4.11$

6.4.17.

(a) **F** (c has to be nonzero, otherwise the set won't be linearly independent; other than that, the statement is true)
(b) **T**(that's the point of the G-S process!)
(c) **T**(Multiply the equation $A = QR$ by Q^T, and you get $Q^T A = Q^T QR = IR = R$, so $R = Q^T A$)

6.4.18.

(a) **T**(I'm assuming $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are obtained using the G-S process)
(b) **T**(In other words, if $\mathbf{x} - \hat{\mathbf{x}} = 0$, then $\mathbf{x}$ is in W)
(c) Ignore (but it's T, and that's because you obtain Q by applying the G-S process to the columns of A)